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STATISTICAL ANALYSES FOR THE WEIBULL DISTRIBUTION WITH EMPHASIS ON CENSORED SAMPLING

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54

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This report comprises three related papers on inferential procedures for the Weibull or extreme-value distribution based on censored samples. In Chapter I a simple, unbiased estimator, based on a censored sample, is proposed for the scale parameter of the extreme-value distribution. The exact distribution of the estimator is determined for the cases in which only the first two or only the first three observations are available. The asymptotic distribution is derived, and an approximate distribution for small sample size is also provided. Interval estimation for the scale parameter is developed and a conservative interval estimate for reliability is also obtained.

Chapter II represents a continuation of the material in Chapter I with emphasis on combining independent lots. A study of the saving in experiment time with censored sampling and a numerical example are also presented.

In Chapter III tables are provided for obtaining confidence limits on the parameters or reliability based on the maximum likelihood estimators for selected censoring and sample sizes.

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FOREWORD

This final report was sponsored by the Aerospace Research Laboratories, Air Force Systems Command, United States Air Force, Wright-Patterson Air Force Base, Ohio, under Contract F 33615-71-C-1040. Work on this contract was technically monitored by Dr. H. Leon Harter under project 7071. Chapters I and II were written by the principal investigator, Dr. Lee J. Bain of the University of Missouri-Rolla, who co-authored Chapter III with Dr. Charles E. Antle and Mr. Barry R. Billman of the Pennsylvania State University.

ABSTRACT

This report comprises three related papers on inferential procedures for the Weibull or extreme-value distribution based on censored samples. In Chapter I a simple, unbiased estimator, based on a censored sample, is proposed for the scale parameter of the extreme-value distribution. The exact distribution of the estimator is determined for the cases in which only the first two or only the first three observations are available. The asymptotic distribution is derived, and an approximate distribution for small sample size is also provided. Interval estimation for the scale parameter is developed and a conservative interval estimate for reliability is also obtained.

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TABLE OF CONTENTS

CHAPTER I

INFERENCES BASED ON CENSORED SAMPLING FROM THE WEIBULL OR EXTREME-VALUE DISTRIBUTION

1. INTRODUCTION

The Weibull distribution denoted by

$$F_X(x) = 1 - \exp\,[-(x/\alpha)^{\beta}],\ 0 < x < \infty;\ \alpha,\beta > 0,$$

is considered in this paper. The variable $Y = \ln X$ follows the extreme-value distribution

$$F_Y(y) = 1 - \exp\,[-\exp[y-u)/b]],\ -\infty < y < \infty,$$

where $b = 1/\beta$ and $u = \ln \alpha$. The variables $(X/\alpha)^{\beta}$ and $(Y-u)/b$ are the corresponding reduced variates whose distributions are given by letting $\alpha = \beta = 1$ and $b = 1$, $u = 0$, respectively. Equivalent procedures can be developed under either model, but the extreme-value distribution has the advantage that its parameters appear as location and scale parameters.

Point and interval estimation procedures are in general quite complicated for these models, especially under censored sampling. Maximum likelihood estimators of the parameters can be determined with the aid of a computer [4,5]; and, inferential procedures based on the maximum likelihood estimators have been rather extensively developed for complete samples [16,17]. Theoretically these methods can be extended to censored samples; however, the amount of computer time needed in order to determine the appropriate distributions becomes excessive. Thus other

estimation and hypothesis testing techniques need to be developed for the censored sampling case.

For point estimation one common approach has been to apply the generalized least squares method or some related method to obtain linear estimators of the location and scale parameters u and b; see for example [9,10,11]. This approach for the most part requires knowledge of the variances and covariances of the ordered observations of the extreme-value distribution, which are available up to n = 25 [13]. It does not appear convenient to determine tests or confidence intervals based on these point estimators for sample sizes larger than 25. Some simple alternate point and interval estimation procedures have been presented in [11,14] for censored sampling. Also, Johns and Lieberman [8] have a notable paper concerned with determining lower bounds for reliability based on censored samples. An attempt is made in this paper to develop procedures which are simple, reasonably good and widely applicable without the necessity of generating an undue number of tables.

2. INFERENCES CONCERNING b

2.1. UNBIASED ESTIMATOR OF b

Suppose $x_1,\ldots,x_r$ denote the r smallest ordered observations in a sample of size n from the Weibull distribution. Also $y_1,\ldots y_r$, where $y_i = \ln x_i$, will represent the r smallest observations in a sample of size n from the extreme-value distribution. The corresponding reduced observations will be denoted by $z_i = (x_i/\alpha)^{\beta}$ and $w_i = (y_i-u)/b$.

On examining a table of coefficients for determining best linear unbiased estimators (BLUE'S) of b, one sees that the statistic

$$\hat{b} = -\sum_{i=1}^{r-1} (Y_i - Y_r)/nk_{r,n} = T/k_{r,n}$$

is an appropriate unbiased estimator of b, where $k_{r,n} = -(1/n)E \sum_{i=1}^{r-1} (W_i - W_r)$. The statistic $\hat{b}$ is the BLUE for r=2, and it is somewhat similar to the BLUE for larger r. The exact moments of the W_i are given in [18] up to n = 100, and $k_{r,n}$ can be calculated easily from these values for any prescribed combination of r and n. For illustration purposes, values of $k_{r,n}$ are presented in Table I for n = 5,10,15,20,30,60,100 and integer values of $r=np\geq 2$ for p = .1(.1)1.0. Asymptotic results can be utilized for larger values of n. If $r/n \to p$ as $n \to \infty$, expressions for the asymptotic values, say k_p, of the constants have been derived, and numerical values of k_p are also presented in Table I. Values of the asymptotic efficiency of $\hat{b}$ are also given in Table I, and these indicate that $\hat{b}$ is a simple, relatively efficient point estimator for b under censored sampling. A comparison of the variance of $\hat{b}$ with the variance of the best linear unbiased estimator of b is also given in Table II for some small sample sizes and various censoring fractions. It may be worth noting that if mean squared error rather than variance is used as a goodness criterion, then a value of c can be found such that the $MSE(c\hat{b})$ is minimized. This value of c is given by

$$c = [1+\mathrm{var}(\hat{b}/b)]^{-1} \doteq nk_{r,n}/(1+nk_{r,n}).$$

A comparison of the mean squared error of $c\hat{b}$ with the mean squared

error of the best linear invariant estimator (BLIE) [9] of b is also given in Table II.

2.2. SUMMARY OF DISTRIBUTION RESULTS

In considering similar tests concerning b relative to the nuisance parameter u, one observes that attention can be restricted to functions of the r - 1 statistics $Y_i - Y_r = \ln(X_i/X_r)$; $i = 1,\ldots,r-1$. The statistic $\hat{b}$ is suggested for use since it is a natural type of function of these statistics to consider, and it is also a good point estimator of b under censored sampling. Another possible indication of its desirability is that, although it is not a sufficient statistic, it does appear as a quantity in the joint density of the $Y_i - Y_r$. Also, since $\hat{b}/b$ is distributed independently of all parameters, tests based on $\hat{b}$ are convenient to express, if the appropriate percentage points are available. The derivation of the distribution of $S = \exp[-nk_{r,n}\hat{b}/b] = \prod_{i=1}^{r-1}(x_i/x_r)^{\beta}$ is considered in section 5.1. For r = 2, the distribution of S is shown to be

$$F_2(s) = ns/(s+n-1),\ 0 < s < 1.$$

For r = 3,

$$F_3(s) = 1 - \frac{n!}{(n-3)!}\left\{\frac{1}{2n} - \frac{1}{2(s+n-1)} - \frac{(n-2)(s-1)}{2q(s+n-1)} - \frac{s}{q^{3/2}}\left[\ln\frac{(n-\sqrt{q})(2s+n-2+\sqrt{q})}{(n+\sqrt{q})(2s+n-2-\sqrt{q})}\right]\right\},$$

where $q = (n-2)^2 - 4s$.

The exact distribution becomes intractable for larger r; however, the chi-square distribution can be used to provide an extremely good approximation. It is shown is section 5.2 that $-2 \ln S = 2nk_{r,n}\hat{b}/b$ is distributed approximately as a chi-square variable with $2nk_{r,n}$ degrees of freedom, for r/n about .5 or less.

The asymptotic distribution of $T/b = -\ln S$ is also derived in section 4. It is shown that the distribution of $\sqrt{n}((T/b) - \mu_p)/\sigma_p$ approaches a standard normal distribution as $n \to \infty$ and $r/n \to p$, where

$$\mu_p = \sum_{i=1}^{\infty} (-\lambda_p)^i/(i)(i)!,$$

$$\sigma_p^2 = p^3/(1-p)\lambda_p^2 - \mu_p^2 + 2\mu_p p/\lambda_p + 2\sum \lambda_p^i(-1)^{i+1}/i^2 i!,$$

and $\lambda_p = -\ln(1-p)$. Numerical values of $k_p = -\mu_p$ and σ_p^2 are tabulated in Table I for p = .1(.1).9.

2.3 TEST OF HYPOTHESIS CONCERNING b AND POWER OF THE TEST

Consider, for example, the test of H_o: $b \le b_o$ against the alternative H_A: $b > b_o$ at the α significance level. Using the chi-square approximation stated in section 2.2, one rejects the hypothesis H_o if

$$-2\sum_{i=1}^{r-1} (Y_i - Y_r)/b_o > \chi_\alpha^2(2nk_{r,n}), \text{ where } \Pr[\chi^2(\nu) > \chi_\alpha^2(\nu)] = \alpha.$$

Linear interpolation can be used for non-integer degrees of freedom.

The power of the test for an alternative b is

$$\Pr[\text{reject } H_o] = P[-2\sum(Y_i - Y_r)/b_o > \chi^2_\alpha(2nk_{r,n})]$$

$$= P[\chi^2(2nk_{r,n}) > (b_o/b)\chi^2_\alpha(2nk_{r,n})]$$

Note that a test on b is analogous to a test on the variance of a normal distribution, since in that case $(n - 1)s^2/\sigma^2$ is distributed as a chi-square variable with n - 1 degrees of freedom. Thus material developed for the normal case can be applied to this case. For example, the sample size table and o.c. curves on pages 299-303 of [2] are applicable by simply replacing σ^2 by b and n by $2nk_{r,n} + 1$.

3. INFERENCES ON THE RELIABILITY

The problem of determining a test for $\xi = \alpha^{1/b}$, or equivalently the reliability, $R = \exp[-(t/\alpha)^\beta]$, will now be considered. It is well known that $2r\hat{\xi} = 2\left[\sum_{i=1}^{r-1} X_i^{1/b} + (n - r + 1)X_r^{1/b}\right]$ is a complete, sufficient statistic for α if b is known, and that $2r\hat{\xi}/\xi$ is distributed as a chi-square variable with 2r degrees of freedom. Since the distribution of $\hat{b}$ is independent of α, it follows from Basu's Theorem [1] that $\hat{\xi}$ and $\hat{b}$ are stochastically independent. Thus a joint confidence region can be determined for b and R, and a conservative limit for R can be obtained. A similar approach has been followed by Mann [9] to obtain a test for R based on X_r/X_1 and $2r\hat{\xi}/\xi$.

In terms of R,

$$2r\hat{\xi}/\xi = 2(-\ln R)[\sum(X_i/t)^{1/b} + (n-r+1)(X_r/t)^{1/b}],$$

and

$$P\left[\frac{2r\hat{\xi}}{\xi} < \chi^2_{\alpha_1}(2r),\ \chi^2_{1-\alpha_2}(2nk_{r,n}) < 2nk_{r,n}\hat{b}/b < \chi^2_{\alpha_3}(2nk_{r,n})\right]$$

$$= (1 - \alpha_1)(1 - \alpha_2 - \alpha_3).$$

This gives the joint confidence region $\{R > \underline{R}(b),\ \underline{b} < b < \bar{b}\}$, where

$$\underline{R}(b) = \exp\{-\chi^2_{\alpha_1}(2r)/2[\sum(X_i/t)^{1/b} + (n-r+1)(X_r/t)^{1/b}]\},$$

$$\underline{b} = 2nk_{r,n}\hat{b}/\chi^2_{\alpha_3}(2nk_{r,n}),\ \text{and}\ \bar{b} = 2nk_{r,n}\hat{b}/\chi^2_{1-\alpha_2}(2nk_{r,n}).$$

A conservative $(1 - \alpha_1)(1 - \alpha_2 - \alpha_3)$ confidence limit for R is then given by $\underline{R} = \min_{\underline{b}<b<\bar{b}} \underline{R}(b)$. It is shown by Mann [9] that $\underline{R}(b)$ is a monotonically decreasing function of b, if the time t is sufficiently small, and at least if $t < (\prod_{i=1}^{r} x_i)^{1/r}$, in which case $\underline{R} = \underline{R}(\bar{b})$. This would ordinarily be the situation if p is not too small, since the expected coverage between any two order statistics is $1/(n+1)$. It is also clear that if $t > x_r$, then $\underline{R}(b)$ is an increasing function of b, and $\underline{R} = \underline{R}(\underline{b})$. If $t < x_r$ but near x_r, then $\underline{R}(b)$ may not be monotonic; but it would have a single minimum, and $\underline{R}(b)$ would approach 1 as $b \to 0$, and $\underline{R}(b)$ would approach $\exp\{-\chi^2_{\alpha_1}(2r)/2n\}$ as $b \to \infty$. Thus, if $t < x_r$ and $\underline{R}(b)$ is decreasing at $\bar{b}$, then $\underline{R} = \underline{R}(\bar{b})$. If $\underline{R}(b)$ is increasing at $\bar{b}$, then a search between $\underline{b}$ and $\bar{b}$ would be needed to determine the minimum.

4. ASYMPTOTIC DISTRIBUTION OF $\hat{b}$

Results given in [3] will be applied to determine the asymptotic distribution of $-T = \sum_{i=1}^{r-1} (Y_i - Y_r)/n$, where $r/n \to p$ as $n \to \infty$. If we follow the notation of [3], $-nT/b = \sum_{i=1}^{r} \ln X_i - r \ln X_r$ $= \sum_{i=1}^{n} c_{in} h(X_i)$, where the X_i are ordered exponential variables. Also, $c_{in} = 1$, $i = 1,\dots,r-1$; $c_{rn} = -r$, $c_{in} = o$, $i = r+1,\dots,n$; $F(x) = 1 - \exp(-x)$, $h(x) = \ln x = \tilde{H}(x)$, $H(u) = h[F^{-1}(u)] = \ln[-\ln(1-u)]$. Also

$$E(X_i) = \sum_{j=1}^{i} 1/(n-j+1),$$

$$\alpha_{jn} = [1/(n-j+1)] \sum_{i=j}^{n} c_{in} \tilde{H}'(E(X_i))$$

$$= (n-j+1)^{-1} [\sum_{i=j}^{r-1} [E(X_i)]^{-1} - r[E(X_r)]^{-1}].$$

Then

$$\mu_n = (1/n) \sum_{j=1}^{n} c_{jn} \tilde{H}[E(X_j)]$$

$$= (1/n) \sum_{j=1}^{r} \ln E(X_j) - (r/n) \ln E(X_r),$$

$$\sigma_n^2 = (1/n) \sum_{j=1}^{r} \alpha_{jn}^2,$$

and the distribution of $\sqrt{n}((-T/b) - \mu_n)/\sigma_n$ approaches a standard normal distribution. Furthermore, $\mu_n \to \mu_p$ and $\sigma_n^2 \to \sigma_p^2$, where $r/n \to p$ as $n \to \infty$. Let

$$J(x) = 1, \quad x \le r/(n+1)$$
$$= 0, \text{ otherwise},$$

and let $a_1 = p$, $\lambda_p = F^{-1}(p) = -\ln(1-p)$. Then

$$\mu_p = \int_0^1 J(u)H(u)du - a_1 h(\lambda_p)$$

$$= \int_0^p \ln[-\ln(1-u)]du - p \ln \lambda_p$$

$$= \sum_{i=1}^{\infty} (-\lambda_p)^i/(i)(i)!.$$

Also,

$$\alpha(u) = (1-u)^{-1}\{\int_u^1 J(w)H'(w)(1-w)dw + a_1(1-p)H'(p)\},$$

and

$$\sigma_p^2 = \int_0^1 [\alpha(u)]^2 du.$$

It can be shown that

$$\sigma_p^2 = p^3/(1-p)\lambda_p^2 - \mu_p^2 + 2\mu_p p/\lambda_p + 2\sum_{i=1}^{\infty} \lambda_p^i(-i)^{i+1}/i^2 i!.$$

Then the distribution of $\sqrt{n}((-T/b) - \mu_p)/\sigma_p$ approaches a standard normal distribution.

Also, $k_{r,n} \to -\mu_p$ as $n \to \infty$; and the asymptotic variance of $\hat{b}$ is $b^2\sigma_p^2/n\mu_p^2$. The asymptotic variance of the maximum likelihood estimator of b is provided in [6] for p = .1(.1).9; and, since this corresponds to the Cramér-Rao lower bound for an unbiased estimator of b, the asymptotic relative efficiency of $\hat{b}$ can be calculated. Some numerical values of $k_p = -\mu_p$, σ_p^2, n var $\hat{b}/b^2$, and the relative efficiency of $\hat{b}$ are presented in Table I.

Note that, for complete samples,

$$k_{n,n} = -E[\sum_{i=1}^{n} (\ln X_i - \ln X_n)]/n$$

$$= \gamma + E \ln X_n$$

$$= \sum_{i=1}^{n} (-1)^i (\ln i) \binom{n}{i},$$

where γ denotes Euler's constant and numerical values of $E(\ln X_n)$ are given in [18]. Also, from [15, page 71] $k_{n,n}$ is approximated by $\gamma + \ln \ln n + \gamma/\ln n$ for large n.

5. DERIVATION OF EXACT AND APPROXIMATE DISTRIBUTIONS

5.1. EXACT DISTRIBUTION FOR r = 2 AND r = 3

The joint density of $(x_1,\ldots,x_r)$ is given by $f(x_1,\ldots,x_r)$

$$= [n!/(n-r)!]\{\prod_{i=1}^{r} (\beta/\alpha)(X_i/\alpha)^{\beta-1} \exp[-(X_i/\alpha)^{\beta}]\} \exp[-(n-r)(X_r/\alpha)^{\beta}],$$

$$0 < x_1 < \ldots < x_r < \infty.$$

On letting $U_i = X_i/X_r$, $i = 1,\ldots,r - 1$, the joint density of the U_i is found to be

$$f(u_1,\ldots,u_{r-1}) = [n!/(n-r)!] \prod_{i=1}^{r-1} \beta u_i^{\beta-1}\Gamma(r)/[\sum_{i=1}^{r-1} U_i^{\beta} + n - r + 1]^r,$$

$$0 < u_1 < \ldots < u_{r-1} < 1,$$

which involves the two quantities $\sum_{i=1}^{r-1} U_i^{\beta}$ and $\prod_{i=1}^{r-1} U_i = \exp[-nk_{r,n}\hat{b}]$ $= S^{1/\beta}$. Since $S^{1/\beta}$ is distributed independently of β, β may be set

equal to one without loss of generality. For $r = 2$, $S = U_1$ and the distribution of S is given directly (see also [7], [11], [12]). For $r = 3$, the distribution of S may be determined by making a change of variable. The work is simplified by noting first that the joint density of U_i's and also the function S are symmetric in the variables. Thus, for $r = 3$,

$$P[S \geq s] = P[U_1U_2 > s], \text{ where } 0 < U_1 < U_2 < 1,$$

$$= \tfrac{1}{2}P[U_1U_2 > s], \text{ where } 0 < U_1 < 1,\ 0 < U_2 < 1.$$

Thus,

$$F_3(s) = 1 - \frac{1}{2}\int_s^1\int_{s/u_2}^1 f(u_1,u_2)du_1du_2.$$

Direct integration yields the result given earlier in the paper. The integration becomes quite tedious for larger values of r, so that an approximation is needed.

5.2. APPROXIMATE DISTRIBUTION

The variable $T = -\sum_{i=1}^{r-1}(Y_i - Y_r)/n$ takes on positive values, and the mean of nT/b is approximately equal to the variance of nT/b, especially for small p. This is verified by Table I for large n, since $\mathrm{var}(nT/b) \doteq n\sigma_p^2$, $E(nT/b) \doteq nk_p$, and $k_p \doteq \sigma_p^2$ in Table I. Thus an approximate distribution with nearly the correct first two moments is obtained by assuming that 2nT/b is distributed approximately as a chi-square variable with $2nk_{r,n}$ degrees of freedom. The approximate percentage points were determined for $r = 2$, $n = 5,10,20$; $r = 3$, $n = 5,10,30$, and $\gamma = .01,.05,.10,.25,$

.50,.75,.90,.95 and .99; and the exact distributions were then used to determine the true probabilities for these approximate percentage points. As seen in Table III, the exact and approximate probabilities are in very close agreement.

This chi-square approximation is also consistent with the asymptotic results, at least to the extent that $k_p \doteq \sigma_p^2$. This follows since $((2nT/b) - 2nk_p)/\sqrt{4nk_p} \doteq (\chi^2(\nu)-\nu)/\sqrt{2\nu}$, which becomes normally distributed as ν increases; but $((2nT/b) - 2nk_p)/\sqrt{4nk_p} \doteq \sqrt{n}((T/b) - k_p)/\sigma_p$ which corresponds to the asymptotic result.

Thus the chi-square approximation seems appropriate if substantial censoring is involved. Further work is needed to determine the amount of error if r/n is near 1.

TABLE I

Values of $k_{r,n}$ such that $E[-\sum_{i=1}^{r-1}(\ln X_i - \ln X_r)/nk_{r,n}] = b$, and asymptotic results for $\hat{b}$ as $r/n \to p$, $n \to \infty$.

	r/n									
n	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
5				.2231		.4813		.8018		1.267
10		.1054	.2172	.3369	.4667	.6098	.7715	.9616	1.202	1.567
15		.1406		.3752		.6536		1.019		1.710
20	.0513	.1583	.2721	.3944	.5277	.6756	.8448	1.048	1.316	1.800
30	.0684	.1759	.2904	.4137	.5482	.6979	.8697	1.077	1.357	1.916
60	.0855	.1936	.3088	.4330	.5687	.7202	.8949	1.108	1.399	2.086
100	.0924	.2007	.3162	.4407	.5770	.7292	.9050	1.120	1.417	2.196
k_p	.1027	.2113	.3272	.4523	.5894	.7427	.9203	1.138	1.444	
σ_p^2	.1027	.2117	.3288	.4568	.6001	.7667	.9725	1.258	1.782	
$nV(\hat{b})/b^2$	9.746	4.742	3.070	2.232	1.728	1.390	1.148	.971	.855	
rel. eff.	.9999	.9994	.9984	.9966	.9934	.9878	.9774	.9559	.8972	

TABLE II

Comparison of $\hat{b}$ with the BLUE and BLIE

n	r	$V(\hat{b}/b)$	V(BLUE/b)	c	$nk_{r,n}/(1+nk_{r,n})$	$MSE(c\hat{b}/b)$	MSE(BLIE/b)
5	3	.4175	.4168	.7055	.7064	.2945	.2942
	4	.2553	.2538	.7966	.8004	.2033	.2024
	5	.1725	.1666	.8529	.8637	.1471	.1428
10	3	.4609	.4607	.6845	.6847	.3155	.3154
	4	.2979	.2975	.7705	.7711	.2295	.2293
	5	.2161	.2155	.8223	.8235	.1777	.1773
	10	.0795	.0716	.9264	.9400	.0736	.0668
20	3	.4809	.4808	.6753	.6753	.3247	.3247
	4	.3162	.3161	.7598	.7599	.2402	.2402
	5	.2338	.2337	.8105	.8107	.1895	.1894
	10	.0960	.0956	.9124	.9134	.0876	.0872

TABLE III

$$\text{Exact } \Pr[2nT/b < \chi^2_{1-\alpha}(2nk_{r,n})]$$

α \ r	2			3		
n	5	10	20	5	10	30
.01	.0098	.0099	.0100	.0097	.0099	.0102
.05	.0501	.0501	.0501	.0502	.0500	.0501
.10	.1012	.1007	.1003	.1017	.1003	.1003
.25	.2541	.2520	.2510	.2538	.2518	.2508
.50	.5020	.5007	.5004	.5039	.5019	.5007
.75	.7417	.7454	.7475	.7469	.7483	.7495
.90	.8864	.8927	.8962	.8926	.8956	.8986
.95	.9374	.9433	.9465	.9426	.9460	.9486
.99	.9833	.9865	.9882	.9859	.9878	.9893

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CHAPTER II

RESULTS FOR ONE OR MORE INDEPENDENT SAMPLES

1. INTRODUCTION

The results presented here represent a continuation of the previous chapter with particular application to a problem considered by Jaech [7].

Jaech [7] develops point and interval estimation procedures for the shape parameter, β, if no more than two failures occur per lot. A means for combining results from two or more lots is also provided if the shape parameters are assumed equal. This problem would be of interest, for example, if groups of items are currently in service for which high reliability is required. Thus, as soon as a few failures occur, the possible necessity of recalling all items for replacement of degradable components would have to be considered.

Procedures for point and interval estimation of β based on two or more failures per lot are presented in the following section with a method for combining results from two or more lots. Point estimation for α and R is considered in section 3 and a conservative lower limit for R is given in section 4. Exact interval estimation procedures for α and R based on two failures per lot are derived in section 5. In section 6 the relative expected experiment time required to obtain a certain precision with censored sampling as compared to complete sampling is studied. A numerical example is given in section 7.

2. ESTIMATION OF β WITH TWO OR MORE FAILURES

In [I] (Chapter 1) an estimator of $b = 1/\beta$ is given as $\hat{b} = -\sum_{i=1}^{r-1} (\ln x_i - \ln x_r)/nk_{r,n} = T/nk_{r,n}$, where the first r failures from n items are observed and the constants $k_{r,n}$ can be obtained from [I] or [11]. (The subscripts on the constants will be suppressed hereafter if the meaning is clear.) Also $2T/b$ is distributed approximately as a chi-square variable with $2nk$ degrees of freedom, at least for r/n less than .5 or so. For $r = 2$ the exact distribution of T/b is given by

$$F(t) = 1 - ne^{-t}/(e^{-t} + n - 1)$$

which is approximately the exponential distribution, $1 - e^{-t}$, for large n. Thus,

$$\begin{aligned} nk_{2,n} &= E(T/b) \\ &= \int_0^\infty [1 - F(t)]\, dt \\ &= n \ln ((n - 1)/n) \doteq 1, \end{aligned}$$

so that $1/\hat{b} \doteq 1/T$, which is the estimator suggested by Jaech [7] for $r = 2$. In this case $\hat{b}$ is the best linear unbiased estimator of b, but as for an exponential variable, $E(1/\hat{b})$ is infinite. This indicates that occasionally very unreasonable estimates will occur if they are based only on a single lot with $r = 2$. A median unbiased estimator, $\hat{\beta}_m = c/T$, can be found by solving $P[c/T < \beta] = \frac{1}{2}$. This gives $c = \ln [(2n - 1)/(n - 1)] \doteq \ln 2$. For $r > 2$, $\hat{\beta}_u = (nk - 1)/(nk\hat{b})$ is approximately an unbiased estimator of β, since if $U \sim \chi^2(\nu)$ then $E(1/U) = 1/(\nu - 2)$ for $\nu > 2$.

Now suppose results are available from two or more lots for which a common value of β is assumed. Suppose there are r_j observed failures from n_j elements in lot j, $j = 1,\ldots,m$. Now the variance of a linear combination, $\sum a_j x_j$, subject to $\sum a_j = 1$ is minimized by choosing $a_j = (1/\sigma_j^2)/\sum (1/\sigma_j^2)$. Since Var $(\hat{b}/b) \doteq 4nk/4n^2k^2 = 1/nk$, this leads to

$$\hat{b}_c = \sum_{j=1}^{m} n_j k_j \hat{b}_j / \sum_{i=1}^{m} n_j k_j = \sum_{j=1}^{m} T_j / \sum_{j=1}^{m} n_j k_j$$

as the combined unbiased estimator of b. Also, $2 \sum_{j=1}^{m} T_j/b$ is distributed approximately as a chi-square variable with $2 \sum_{j=1}^{m} n_j k_j$ degrees of freedom; thus confidence limits or tests concerning b or β are immediately available based on one or more lots.

3. POINT ESTIMATION FOR α AND R

A closed form estimator of α is given by $\hat{\alpha} = [(\sum_{i=1}^{r-1} x_i^{\hat{\beta}} + (n - r + 1)x_r^{\hat{\beta}})/r]^{1/\hat{\beta}}$. This is in the form of the usual maximum likelihood estimator [3] of α with the maximum likelihood estimator of β being replaced by the simple closed form estimator being considered in this paper. Similarly a closed form estimator of reliability is given by $\hat{R} = \exp[-(t/\hat{\alpha})^{\hat{\beta}}]$. The results of a Monte Carlo study of the means and variances of these estimators are presented in Table 1. The tabled values are based on 2,000 samples generated from a Weibull distribution with $\alpha = 1$ and $\beta = 1$. Corresponding values for maximum likelihood estimators are available for some cases [2,5] and these are included for comparison purposes. The results of course are applicable for other values of the parameters to the extent that for both methods of estimation $\hat{\beta}/\beta$ and $(\hat{\alpha}/\alpha)^{b}$ are pivotal quantities with distributions independent of both parameters. Except for the simple estimators $\hat{b}$ and $\hat{\beta}_u$ there appears to be substantial bias for small r with both methods of estimation. An unbiased estimator of $u = \ln \alpha$ could be obtained for a given n and r by use of Monte Carlo work if this were deemed to be worthwhile. For example, $E[\beta \ln (\hat{\alpha}/\alpha)] = E(\ln \hat{\alpha}_{11})$ where $\hat{\alpha}_{11}$ denotes the estimator calculated from samples generated with $\alpha = 1$ and $\beta = 1$. Thus $\ln \hat{\alpha} - \hat{b}\, E(\ln \hat{\alpha}_{11})$ is an unbiased estimator of $\ln \alpha$.

To obtain a combined estimator from two or more lots consider the following. Let $\xi = \alpha^{\beta}$ and $\hat{\xi}(\beta) = [\sum_{i=1}^{r-1} x_i^{\beta} + (n - r + 1)x_r^{\beta}]/r$. It is well known that $2r\hat{\xi}/\xi \sim \chi^2(2r)$. Thus if β were known,

$\sum_{j=1}^{m} r_j \hat{\xi}_j / \sum_{j=1}^{m} r_j$ would be the appropriate linear combination of the estimators to use to obtain a combined estimator of ξ with minimum variance. Since $\hat{\alpha}^{\hat{\beta}} = \hat{\xi}(\hat{\beta})$, this suggests using $\hat{\xi}_c = \sum_{j=1}^{m} r_j \hat{\alpha}_j^{\hat{\beta}_j} / \sum_{j=1}^{m} r_j$ as the combined estimator of ξ. Also $\hat{\alpha}_c = \hat{\xi}_c^{\hat{b}_c}$ and $R_c = \exp\,[-(t/\hat{\alpha}_c)^{\hat{\beta}_c}]$.

4. INTERVAL ESTIMATION OF R

Joint confidence intervals for b and R and conservative lower limits for R are given in [I] for a single lot. Similar results can be obtained for combined lots. As mentioned earlier, $2\sum_{j=1}^{m} \hat{\xi}_j(\beta)/\xi \sim \chi^2(2\sum_{j=1}^{m} r_j)$ and $2\sum_{j=1}^{m} T_j/b$ is distributed approximately as $\chi^2(2\sum_{j=1}^{m} n_j k_j)$. Also $\hat{b}_j$ and $\hat{\xi}_j(\beta)$ are independent so

$$P[2\sum r_j \hat{\xi}_j(\beta)/\xi < \chi_{\delta_1}^2(2\sum r_j), \chi_{1-\delta_2}^2(2\sum n_j k_j) < 2\sum T_j/b < \chi_{\delta_3}^2(2\sum n_j k_j)]$$

$$= (1 - \delta_1)(1 - \delta_2 - \delta_3),$$

where $P[\chi^2(\nu) > \chi_\delta^2(\nu)] = \delta$. This gives the joint confidence region $\{R > \underline{R}(b), \underline{b} > b > \bar{b}\}$, where

$$\underline{R}(b) = \exp\{-t^\beta \chi_{\delta_1}^2(2\sum r_j)/2\sum r_j \hat{\xi}_j(\beta)\},$$

$$\underline{b} = 2\sum T_j/\chi_{\delta_3}^2(2\sum n_j k_j),$$

$$\bar{b} = 2\sum T_j/\chi_{1-\delta_2}^2(2\sum n_j k_j).$$

A conservative $(1 - \delta_1)(1 - \delta_2 - \delta_3)$ lower confidence limit for R is given by $\underline{R} = \min_{\underline{b} \le b \le \bar{b}} \underline{R}(b)$. As for a single lot, $\underline{R}(b)$ is either a monotonic function of b or has a single minimum. In particular if t is sufficiently small and at least if $\sum_{j=1}^{m} r_j \ln(t/x_{r_j}) < -\sum_{j=1}^{m} T_j$, then $\underline{R}(b)$ is a monotonically decreasing function of b and $\underline{R} = \underline{R}(\bar{b})$. Also if $t > \max_j x_{r_j}$, then $\underline{R} = \underline{R}(\underline{b})$.

It is theoretically possible to determine exact confidence limits for R based on $\hat{R}$ and exact confidence limits for α based on

the pivotal quantity $(\hat{\alpha}/\alpha)^{\hat{\beta}}$; however, it has not been possible in general to determine the necessary distributions mathematically. Some Monte Carlo work has been carried out for these quantities using the less convenient maximum likelihood estimators [2], and tables are given in [8] for obtaining confidence limits for u or b and tolerance bounds for the distribution based on the best linear invariant estimators of u and b for n = 3(1)25, r = 3(1)n. Exact results for the case r = 2 are considered in the next section.

5. INTERVAL ESTIMATION FOR α AND R BASED ON TWO FAILURES PER LOT

For $r = 2$ it can be shown that $\hat{R}$ is a monotonically increasing function of

$$v = \ln (x_2/t)/\ln (x_2/x_1) = \ln [(x_2/t)^\beta]/[\ln (x_2/t)^\beta - \ln (x_1/t)^\beta],$$

the distribution of which depends only on $\theta = (\alpha/t)^\beta = -1/\ln R$. Letting $Z_i = (X_i/t)^\beta$, then we have

$$f(z_1,z_2) = [n(n - 1)/\theta^2] \exp [-(z_1/\theta) - (n - 1)z_2/\theta],$$
$$0 < z_1 < z_2 < \infty.$$

Making the transformation $Y_1 = Z_1/Z_2$, $Y_2 = Z_2$ leads to

$$f(y_1,y_2) = (y_2/\theta^2) \exp [-(y_1y_2/\theta) - (n - 1)y_2/\theta];$$
$$0 < y_1 < 1,\ 0 < y_2 < \infty.$$

Thus,

$$F(v) = P[-\ln Y_2/\ln Y_1 < v]$$
$$= P[Y_2 < Y_1^{-v}].$$

For $v = 0$, $$F_\theta(0) = \int_0^1 \int_0^1 f(y_1,y_2)\, dy_1\, dy_2$$
$$= 1 - ne^{-(n-1)\theta} + (n - 1)e^{-n/\theta}.$$

For $v < 0$, $$F_\theta(v) = \int_0^1 \int_{y_2^{-1/v}}^1 f(y_1,y_2)\, dy_1\, dy_2$$
$$= F_\theta(0) - \int_0^1 \frac{n(n - 1)}{\theta} e^{-(n-1)y_2/\theta}(1 - e^{-y_2^{1-1/v}/\theta})\, dy_2.$$

For $v > 0$, $$F_\theta(v) = F_\theta(0) + \int_1^\infty \frac{n(n - 1)}{\theta} e^{-(n-1)y_2/\theta}(1 - e^{-y_2^{1-1/v}/\theta})\, dy_2.$$

Also, $F_0(1) = 1 - \exp(-n/\theta)$.

The above integrals were evaluated by numerical integration and $F_\theta(v)$ is tabulated in Table 2. Since $F_\theta(v)$ depends primarily on $(n - 1)/\theta$ rather than on n and θ separately, $F_\theta(v)$ was tabulated as a function of $(n - 1)/\theta$ and v for n = 10, 20, and 100, and linear harmonic interpolation should be accurate for other values of n. For an observed sample value v, a lower $1 - \delta$ confidence limit, $\underline{\theta}$, for θ is the value of θ such that $F_\theta(v) = 1 - \delta$, also $\underline{R} = \exp(-1/\underline{\theta})$. For example suppose a 90% lower confidence limit for θ is desired and for n = 50, v = .3 is observed. From Table 2 $(n - 1)/\underline{\theta} \doteq 3$ and $\underline{\theta} \doteq 16.3$, also $\underline{R} \doteq .94$. Of course with only two observed failures these methods are not likely to be precise enough to be of practical value unless results from more than one lot are available.

Although no special techniques have been developed for combining lots in this case, standard general procedures can be used. For example $-2 \ln F_\theta(V) \sim \chi^2(2)$ so for m independent lots, $-2 \sum_{i=1}^{m} \ln F_\theta(V_i) \sim \chi^2(2m)$. For observed values $v_1, \ldots, v_m$, the lower $1 - \delta$ confidence limit for θ is the value $\underline{\theta}$ which satisfies $-2 \sum_{i=1}^{m} \ln F_{\underline{\theta}}(v_i) = \chi^2_{1-\delta}(2m)$, where $P[\chi^2(\nu) > \chi_\delta^2(\nu)] = \delta$. The hypothesis testing format is somewhat more convenient in this case. One would reject $H_o: \theta \leq \theta_o$ in favor of the alternative $H_a: \theta > \theta_o$ if $-2 \sum_{i=1}^{m} \ln F_{\theta_o}(v_i) < \chi^2_{1-\delta}(2m)$.

The above results can also be used to determine confidence limits for α by letting $t = \alpha$ and $\theta = 1$. That is, $F_1(v)$ is the cumulative distribution function for $V = \ln(X_2/\alpha)/\ln(X_2/X_1)$. If

$F_1(v_\gamma) = \gamma$, then

$$P[V < v_\gamma] = P[\alpha > x_2(x_1/x_2)^{v_\gamma}] = \gamma,$$

and $x_2(x_1/x_2)^{v_\gamma}$ is a lower $100\gamma\%$ confidence limit for α. Probabilities for this case are included in Table 2 under $(n - 1)/\theta = n - 1$.

6. COMPARISON OF CENSORED SAMPLING AND COMPLETE SAMPLING

Consider a test of H_o: $\beta \leq \beta_o$ against H_a: $\beta > \beta_o$ at the δ level of significance. A test for this hypothesis has been developed in [9] for complete samples based on the maximum likelihood estimator, say $\tilde{\beta}$. The null hypothesis is rejected if $\tilde{\beta} > \beta_o \ell_{1-\delta}$, where ℓ_γ satisfies $P[\tilde{\beta}/\beta > \ell_\gamma] = \gamma$ and is tabulated in [9]. The power of the test for the alternative β_a is given by $P[\tilde{\beta}/\beta > (\beta_o/\beta_a)\ell_{1-\delta}]$. If we use the simple censored sample estimator the null hypothesis is rejected if $\hat{b} < (1/\beta_o)c_\delta(\nu)$, where $\nu = 2nk_{r,n}$ and $c(\nu)$ denotes the chi-square over degrees of freedom distribution [1,4]. The power is given by $P[c_\nu < (\beta_a/\beta_o)c_\alpha(\nu)]$. Values of ν are given in Table 3 which would provide the same power for the censored sampling test as would be obtained by using the maximum likelihood method based on a complete sample of size N. Some combinations of r and n which would yield these values of ν are also given and the relative expected experiment time, R.E.T. = $E(X_{r,n})/E(X_{N,N})$, is given in each case for $\beta = 1$ and $\beta = 2$. Note that the table indicates that the comparison does not appear to be very sensitive to the level of significance or the level of the power being considered.

7. NUMERICAL EXAMPLE

Harter and Moore [6] give a simulated sample of size 40 from a Weibull population with shape parameter 2, scale parameter 100 and location parameter 10; and, they calculated maximum likelihood estimates based on the smallest 10, 20, 30 and 40 observations respectively. After subtracting the value of the location parameter from each observation we calculated numerical values which are given in Table 4 of the following quantities for $r = 2, 10, 20, 30$ and 40:

$\nu = 2nk_{r,n}$, $\hat{b}$, $\hat{\beta} = 1/\hat{b}$, $\hat{\beta}_u = ((\nu - 2)/\nu)\hat{\beta}$, $\underline{\beta}(.025) = \chi^2_{.975}(\nu)/(\nu\hat{b}) = C_{.975}(\nu)\hat{\beta}$, and $\overline{\beta}(.025) = C_{.025}(\nu)\hat{\beta}$, where $\underline{\beta}(\delta)$ denotes a lower $1 - \delta$ confidence limit for β. The lower limit based on maximum likelihood estimates will be denoted by $\underline{\beta}_m(\delta)$.

The hypothesis H_o: $\beta = \beta_o = 1$ is rejected in favor of the alternative H_A: $\beta > 1$ at the δ significance level if $\hat{b}\beta_o < C_{1-\delta}(\nu)$. The level of significance, δ', at which this hypothesis could have been rejected is given in Table 4. The power of a δ level test of H_o against an alternative β_a is denoted by $P(\delta,\beta_a) = P[C_\nu < \beta_a C_{1-\delta}(\nu)]$ in Table 4. In this example the true reliability is .90 at $t = 32.46$. A conservative .9025 lower confidence limit for R, $\underline{R} = \min\limits_{\underline{b}(.025)<b<\overline{b}(.025)} \underline{R}(b)$, where $\underline{R}(b) = \exp\{-\chi^2_{.05}(2r)/2[\sum_{i=1}^{r-1}(x_i/t)^{\beta} + (n - r + 1)(x_r/t)^{\beta}]\}$, was determined for $t = 32.46$. For $r = 20, 30$ and 40 the condition $\ln t - \ln x_r < -T/r$ holds so that

$\underline{R} = \underline{R}(\bar{b}(.025))$. This is also the limit for r = 10 although the above condition does not hold in this case.

The maximum likelihood estimates were included in the table for comparison. Also confidence bounds for b and R are available from [10] for complete sampling and from [III] (Chapter III) or [2] for 25% and 50% censoring and these are included. Although confidence bounds are included for the complete sample case for comparison purposes, it should be recalled that the efficiency of $\hat{b}$ and the accuracy of the chi-square approximation are not as great for large r/n. An estimate of a lower bound for reliability, $\underline{R}(\hat{b})$, would perhaps also be a useful statistic and it is included in the table.

TABLE 1

Monte Carlo study of means and variances of estimators; $\alpha = 1$, $\beta = 1$. (Tildes refer to maximum likelihood estimators and carets to the simple estimators).

n	20					40				
r	2	5	10	15	20	2	10	20	30	40
$E(\hat{b})$	.986	.995	.997	.998	1.003	.980	1.000	1.000	1.002	1.004
$E(\tilde{b})$	.51	.81	.91	.95	.96					
$E(\hat{\beta})$		1.31	1.11	1.06	1.04		1.11	1.046	1.026	1.018
$E(\tilde{\beta})$					1.07			1.098	1.060	1.036
$E(\hat{\beta}_u)$		1.01	1.00	1.01	1.01		.999	.999	1.000	1.005
$E(\hat{\alpha})$		1.24	1.007	.992	.995		1.10	1.000	.994	.998
$E(\hat{u})$		-.12	-.060	-.043	-.034		-.06	-.032	-.023	-.017
$E(\tilde{u})$		-.34	-.11	-.04	-.02					
$E\hat{R}(.75)$	.46	.69	.738	.744	.744	.42	.72	.743	.746	.747
$E\tilde{R}(.75)$					.756			.752	.755	.753
$E\hat{R}(.90)$	.73	.89	.894	.894	.892	.63	.89	.896	.896	.896
$E\tilde{R}(.90)$					.902			.904	.903	.901
$E\hat{R}(.95)$	.87	.94	.944	.944	.943	.80	.95	.947	.947	.946
$E\tilde{R}(.95)$					.950			.951	.951	.950
$V(\hat{b})$	.93	.218	.095	.057	.042	.92	.102	.044	.027	.022
$V(\tilde{b})$	.23	.161	.081	.051	.032					
$V(\tilde{b}_u)$	.89	.245	.098	.057	.035					
$V(\hat{\beta})$		.922	.145	.074	.051		.162	.054	.030	.023
$V(\tilde{\beta})$					.036			.063	.033	.019
$V(\hat{\alpha})$		1.780	.140	.067	.057		.580	.067	.033	.028
$V(\hat{u})$		.615	.136	.070	.059		.290	.066	.034	.029
$V(\tilde{u})$		.48	.131	.07	.056					
$V(\hat{R}(.75))$	.133	.030	.008	.0068	.0072	.148	.009	.0035	.0033	.0037
$V(\tilde{R}(.75))$					.0062			.0040	.0035	.0031
$V(\hat{R}(.90))$	.105	.007	.003	.0029	.0030	.145	.002	.0016	.0015	.0016
$V(\tilde{R}(.90))$					.0023			.0016	.0015	.0012
$V(\hat{R}(.95))$	.052	.002	.0015	.0013	.0013	.097	.001	.0007	.0007	.0007
$V(\tilde{R}(.95))$					.0009			.0008	.0007	.0005

TABLE 2

Values of $P[\ln (X_2/t)/\ln (X_2/X_1) < \nu]$, $\theta = (\alpha/t)^{\beta} = -1/\ln R$.

$n = 10$

v \ $\frac{n-1}{\theta}$	1	2	3	4	5	6	7	8	9
-30.0	.006	.015	.024	.032	.038	.043	.047	.051	.055
-20.0	.009	.023	.036	.047	.056	.063	.070	.075	.080
-10.0	.017	.044	.069	.090	.107	.122	.134	.144	.153
- 9.0	.018	.049	.076	.100	.118	.134	.147	.158	.168
- 8.0	.020	.054	.085	.111	.132	.149	.163	.176	.187
- 7.0	.023	.061	.096	.125	.149	.168	.184	.198	.210
- 6.0	.027	.070	.111	.144	.170	.192	.210	.226	.239
- 5.0	.031	.083	.130	.168	.199	.225	.246	.263	.279
- 4.0	.038	.101	.157	.204	.240	.270	.295	.316	.334
- 3.5	.043	.113	.176	.227	.268	.301	.328	.350	.370
- 3.0	.049	.128	.200	.257	.302	.339	.368	.393	.414
- 2.5	.057	.149	.230	.296	.347	.387	.420	.448	.471
- 2.0	.068	.177	.272	.348	.406	.452	.489	.519	.544
- 1.5	.084	.217	.332	.421	.489	.540	.581	.614	.641
- 1.0	.111	.281	.424	.531	.609	.666	.709	.743	.769
- 0.5	.161	.396	.579	.704	.787	.842	.879	.905	.923
- 0.3	.196	.469	.669	.796	.872	.918	.945	.963	.974
- 0.1	.248	.566	.773	.887	.945	.974	.987	.994	.997
0.0	.284	.622	.823	.923	.967	.987	.995	.998	.999
0.1	.328	.678	.864	.946	.979	.992	.997	.999	1.000
0.3	.427	.766	.913	.969	.989	.996	.999	1.000	
0.6	.557	.842	.946	.982	.994	.998	.999		
1.0	.671	.892	.964	.988	.996	.999	1.000		

TABLE 2 (continued)

n = 20

v \ $\frac{n-1}{\theta}$	1	2	3	4	5	6	7	8	9	19
-30.0	.006	.015	.024	.032	.038	.044	.048	.052	.056	.086
-20.0	.008	.022	.036	.047	.057	.064	.071	.077	.082	.121
-10.0	.016	.043	.069	.091	.109	.123	.136	.147	.156	.218
- 9.0	.018	.048	.076	.100	.120	.136	.150	.161	.172	.237
- 8.0	.020	.053	.085	.111	.133	.151	.166	.179	.190	.262
- 7.0	.022	.060	.096	.126	.150	.170	.187	.201	.214	.291
- 6.0	.026	.069	.110	.144	.172	.195	.214	.230	.244	.329
- 5.0	.030	.082	.129	.169	.201	.228	.249	.268	.284	.379
- 4.0	.037	.099	.157	.204	.242	.273	.299	.321	.339	.446
- 3.5	.042	.111	.175	.228	.270	.304	.332	.356	.376	.489
- 3.0	.047	.126	.199	.257	.304	.342	.373	.399	.421	.541
- 2.5	.055	.146	.229	.296	.349	.391	.425	.453	.477	.605
- 2.0	.066	.173	.270	.348	.408	.455	.493	.524	.550	.684
- 1.5	.082	.213	.330	.421	.490	.544	.585	.619	.646	.780
- 1.0	.107	.276	.420	.529	.609	.668	.713	.747	.774	.890
- 0.5	.155	.388	.572	.700	.785	.842	.880	.906	.925	.983
- 0.3	.189	.459	.661	.790	.869	.916	.945	.963	.974	.998
- 0.1	.239	.553	.763	.881	.941	.972	.986	.993	.997	1.000
0.0	.274	.608	.812	.916	.964	.985	.994	.997	.999	
0.1	.316	.662	.852	.939	.976	.991	.996	.999	1.000	
0.3	.412	.749	.902	.963	.986	.995	.998	.999		
0.6	.538	.826	.937	.977	.992	.997	.999	1.000		
1.0	.651	.878	.957	.985	.995	.998	.999			

TABLE 2 (continued)

n = 100

v \ $\frac{n-1}{\theta}$	1	2	3	4	5	6	7	9	19	99
-30.0	.005	.015	.024	.032	.039	.044	.049	.057	.088	.150
-20.0	.008	.022	.036	.047	.057	.065	.072	.084	.123	.204
-10.0	.016	.043	.069	.091	.109	.125	.138	.159	.222	.347
- 9.0	.017	.047	.076	.100	.120	.137	.151	.174	.242	.375
- 8.0	.019	.053	.085	.112	.134	.153	.168	.193	.266	.408
- 7.0	.022	.060	.095	.126	.151	.172	.189	.217	.297	.449
- 6.0	.025	.068	.110	.144	.173	.197	.216	.248	.335	.498
- 5.0	.030	.081	.129	.169	.202	.230	.252	.288	.385	.560
- 4.0	.036	.098	.156	.204	.244	.276	.302	.344	.452	.638
- 3.5	.041	.110	.174	.228	.271	.306	.335	.380	.496	.685
- 3.0	.046	.125	.197	.258	.306	.345	.376	.425	.548	.739
- 2.5	.054	.144	.228	.296	.350	.394	.429	.482	.612	.798
- 2.0	.064	.171	.269	.348	.409	.458	.497	.555	.690	.863
- 1.5	.079	.210	.327	.420	.491	.546	.589	.651	.785	.927
- 1.0	.104	.272	.417	.528	.609	.670	.715	.777	.893	.979
- 0.5	.151	.381	.567	.697	.784	.841	.880	.926	.983	.999
- 0.3	.184	.451	.654	.786	.867	.915	.945	.974	.998	1.000
- 0.1	.233	.543	.755	.875	.938	.970	.985	.997	1.000	
0.0	.266	.597	.803	.910	.960	.983	.993	.999		
0.1	.307	.650	.843	.934	.973	.989	.996	.999		
0.3	.400	.736	.893	.958	.984	.994	.998	1.000		
0.6	.524	.813	.929	.974	.990	.996	.999			
1.0	.636	.867	.952	.982	.994	.998	.999			

TABLE 3

Comparison between maximum likelihood test (complete sample) and simple censored sample test.

N	α	β_a/β_o	Power	ν	n	15	20	30	60	100
10	.10	1.73	.75	28	r	12	13	14	14	14
10	.10	2.00	.90	28	r	12	13	14	14	14
10	.05	2.25	.90	28	r	12	13	14	14	14
			R.E.T. (β=1)			.51	.34	.21	.09	.05
			R.E.T. (β=2)			.72	.59	.46	.30	.23
20	.05	1.73	.90	59	r			23	27	29
20	.05	1.83	.95	60	r			23	27	29
			R.E.T. (β=1)					.39	.16	.09
			R.E.T. (β=2)					.63	.41	.31

TABLE 4

Numerical example, $n = 40$, $\beta = 2$, $\alpha = 100$.

r	ν	$\hat{b}$	$\tilde{b}$	$\hat{\beta}$	$\tilde{\beta}$	$\hat{\beta}_u$	$\tilde{\beta}_u$	$\underline{\beta}(.025)$	$\underline{\beta}_m(.025)$	$\overline{\beta}(.025)$	$\overline{\beta}_m(.025)$
2	2.03	.68	.34	1.46	2.90			.04		5.36	
10	19.30	.81	.73	1.24	1.37	1.11		.58		2.13	
20	44.68	.48	.48	2.08	2.09	1.99	1.90	1.31	1.19	3.02	2.92
30	78.50	.58	.56	1.73	1.78	1.69	1.68	1.23	1.18	2.31	2.36
40	159.21	.53	.51	1.88	1.95	1.86	1.88	1.49	1.45	2.32	2.41

r	$\hat{\alpha}$	$\tilde{\alpha}$	$\hat{R}(32.46)$	$\tilde{R}(32.46)$	$\underline{R}$	$\underline{R}_m$	$\underline{R}(\hat{b})$	δ'	P(.025,1.5)	P(.025,2.0)
2	76.5	27.9	.75	.64						
10	151.3	136.6	.86	.87	.73		.79	.30	.19	.47
20	83.9	83.8	.87	.87	.72	.79	.82	.005	.43	.88
30	96.4	96.3	.86	.87	.72	.79	.82	.001	.68	.99
40	92.2	92.8	.87	.88	.75	.82	.84	<.001	.95	>.999

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CHAPTER III

RESULTS FOR CENSORED SAMPLING BASED ON THE MAXIMUM LIKELIHOOD ESTIMATORS

1. INTRODUCTION AND NOTATION

In life testing experiments it is a fairly common practice to terminate the experiment before all items have failed. The Weibull distribution is often used as a model for the observations and when a computer is available maximum likelihood estimation of the parameters is to be recommended. The tables presented in this paper enable one to set confidence limits on the parameters and the reliability based on the maximum likelihood estimates for selected censoring and sample sizes.

It is also observed that, as in the case with no censoring, the maximum likelihood estimator of the reliability is very nearly unbiased and its variance is near the Cramér-Rao lower bound. Unbiasing factors for the maximum likelihood estimator of the shape parameter are given.

The form of the Weibull distribution function considered in this chapter is

$$F(t;b,c) = 1 - \exp\left(-(t/b)^c\right) \text{ for } t \geq 0$$

where b is the scale parameter and c is the shape parameter The

reliability at time t is simply $R(t) = \exp(-(t/b)^c)$.

Let $\hat{b}$ and $\hat{c}$ be the maximum likelihood estimators of b and c and let $\hat{R}(t)$ be the maximum likelihood estimator of R(t). Then it is known [1,5] that the distribution of $\hat{c}/c$ and $\hat{c} \log (\hat{b}/b)$ does not depend upon b and c, although it will, of course, depend upon the sample size, n, and the number of observations before censoring, r. Thus for a given n and r these pivotal functions can be used to test hypotheses about b and c or set confidence intervals on b and c. The tables required when there is no censoring are given by Thoman, Bain and Antle [4], and this paper presents the tables when either 25% or 50% of the largest observations are censored. Moreover, it appears that linear interpolation should be adequate for censoring levels between those given in the tables.

It was shown [5] that the distribution of $\hat{R}(t)$ depends only upon the values of R(t), n and r. Tables providing lower confidence limits for R(t) based upon m.l.e.'s from complete samples are given by Thoman, Bain and Antle [5], and this paper presents the tables needed when either 25% or 50% of the sample values are censored from above.

The values for each n were obtained by simulation with 8000 samples (of size n) used for n = 40, 60, 80, 100 and 120. The 8000 samples were run in two sets of 4000 and the critical values for each set of 4000 were compared. The critical values for the reliability tables for $\hat{R}(t) \geq .9$ usually differed by less than .004, and so we believe there is little sampling error in these tables. The critical values for the other tables differed somewhat

more, those for γ's of .05, .1, .9 and .95 usually differed by about .02.

2. Inferences on the Parameters

2.1 Inferences on the shape parameter

The standardized function $\sqrt{n}\,(\hat{c}/c - E(\hat{c}/c))$ was considered in this case because of the convenience in the use of the asymptotic values and for better interpolation in the table. Table 1 gives percentage points for this quantity for selected cumulative probability levels. The asymptotic percentage points were obtained from the work of Harter and Moore [2] and are also included in the table. It is seen from the table that the asymptotic values are approached quite slowly. We believe this is due to the lack of symmetry when the samples are censored on one side, and it appears that the asymptotic values are not very useful in the censored case. Unbiasing factors for $\hat{c}$ are included in Table 1.

Tests of hypotheses concerning c or confidence intervals for c based on the function $\sqrt{n}\,(\hat{c}/c - E(\hat{c}/c))$ can be easily developed with the aid of Table 1. For example a $1 - \alpha$ confidence interval for c is given by

$$[\hat{c}/(E(\hat{c}/c) + z_{1-\alpha/2}/\sqrt{n}), \hat{c}/(E(\hat{c}/c) + z_{\alpha/2}/\sqrt{n})],$$

where the z_γ and $E(\hat{c}/c)$ are given in Table 1.

2.2 Inferences on the scale parameter

Again as an aid in interpolation, percentage points for the expression $\sqrt{n}\,\hat{c}\,\ln(\hat{b}/b)$ are given in Table 2. Interpolation

should be fairly good, but as was true for the shape parameter it appears that the approach to the asymptotic values is quite slow, and for one sided censored samples with n less than 120 the asymptotic values should not be used. In this case, for example, a $100(1 - \alpha)\%$ confidence interval for b is given by $[\hat{b} \exp (-u_{1-\alpha/2}/\sqrt{n\hat{c}}), \hat{b} \exp (-u_{\alpha/2}/\sqrt{n\hat{c}})]$.

3. Inferences on the Reliability

In many studies in which the Weibull distribution is used as a model, the primary interest is in the reliability at some time t, R(t). It is fortunate that in spite of skewness, censoring and other difficulties, the m.l.e. of R(t) for reasonable values of R(t) has negligible bias and its variance is very close to the Cramér-Rao lower bound for the variance of an unbiased estimator of R(t). This property was noted in [5] for complete sampling, and it also holds for censored sampling. The bias of $\hat{R}(t)$ is given in Table 3 and the variance in Table 4. A comparison of the variances and the Cramér-Rao lower bounds is given in Table 5.

Table 6 gives lower confidence limits on R(t). These are read directly from the table by entering the value of $\hat{R}(t)$ observed. A need for these tables to include high reliability levels has been communicated to the authors, and this accounts for the number of entries for reliabilities near 1 in the tables.

4. Example

Harter and Moore [3] give a simulated sample of size 40 from a Weibull population with shape parameter 2, scale parameter 100

and location parameter 10; and, they calculated maximum likelihood estimates based on the smallest 10, 20, 30 and 40 observations, respectively. This example with the location parameter assumed known may be used to illustrate the use of the tables.

For $r = 20$, $\hat{c} = 2.091$ and $\hat{b} = 83.8$. The unbiased estimate of c is $(.911)(2.091) = 1.90$. Also for example, a test of H_o: $c = 1$ against the alternative H_A: $c > 1$ corresponds to a test of whether an exponential model is appropriate, or whether a Weibull model with an increasing failure rate is needed. This hypothesis is rejected at the .05 level if $\sqrt{40}\ (\hat{c}/1 - 1.098) > 2.95$, or if $\hat{c} > 1.56$. Thus the hypothesis is rejected. A 90% confidence interval for c is given by

$$[2.091/(1.098 + 2.95/\sqrt{40}),\ 2.091/(1.098 - 2.09/\sqrt{40})]$$
$$= [1.34,\ 2.72].$$

A 90% confidence interval for b is

$$[83.8 \exp\ (-2.16/(2.091)\sqrt{40}),\ 83.8 \exp\ (3.75/(2.091)\sqrt{40})]$$
$$= [71.17,\ 111.27].$$

In this example the true reliability at $t = 32.459$ is .90. The m.l.e. for $r = 20$ is $\hat{R} = \exp\ (-(32.459/83.8)^{2.091}) = .871$. From Table 6 a lower 90% confidence limit for R(t) is .80.

TABLE 1

Percentage points, z_γ, such that $P\{\sqrt{n}(\hat{c}/c - E(\hat{c}/c)) < z_\gamma\} = \gamma$, $T = \sqrt{n}(\hat{c}/c)$

n	$\frac{r}{n}$ \ γ	.01	.05	.1	.9	.95	.99	V(T)	$E(\hat{c}/c)$	$1/E(\hat{c}/c)$
40	1.00	-1.60	-1.20	- .98	1.10	1.60	2.41	.74	1.036	.965
	.75	-2.15	-1.56	-1.30	1.59	2.09	3.30	1.33	1.060	.942
	.50	-2.62	-2.09	-1.74	2.09	2.95	4.90	2.52	1.098	.911
60	1.00	-1.59	-1.21	- .97	1.05	1.43	2.18	.67	1.024	.976
	.75	-2.03	-1.50	-1.19	1.48	2.03	3.11	1.18	1.036	.965
	.50	-2.62	-2.02	-1.62	1.96	2.70	4.28	2.10	1.060	.942
80	1.00	-1.62	-1.20	- .97	1.04	1.40	2.10	.66	1.019	.981
	.75	-2.02	-1.51	-1.21	1.49	1.99	3.06	1.18	1.027	.974
	.50	-2.60	-1.98	-1.61	1.99	2.74	4.22	2.13	1.047	.953
100	1.00	-1.63	-1.20	- .95	1.04	1.36	2.06	.67	1.016	.984
	.75	-1.99	-1.46	-1.16	1.46	1.92	2.90	1.10	1.022	.978
	.50	-2.54	-1.93	-1.54	1.96	2.59	4.00	1.97	1.035	.966
120	1.00	-1.64	-1.26	- .97	.99	1.33	2.06	.66	1.012	.988
	.75	-2.05	-1.48	-1.15	1.45	1.97	2.88	1.10	1.018	.982
	.50	-2.61	-1.93	-1.55	1.89	2.51	3.81	1.90	1.030	.971
∞	1.00	-1.81	-1.28	- .99	.99	1.28	1.81	.61	1.0	1.0
	.75	-2.35	-1.66	-1.29	1.29	1.66	2.35	1.02	1.0	1.0
	.50	-3.05	-2.15	-1.68	1.68	2.15	3.05	1.72	1.0	1.0

TABLE 2

Percentage points, u_γ, such that $P(\sqrt{n}\hat{c}\ \ln(\hat{b}/b) < u_\gamma) = \gamma$

n	$\frac{r}{n}$ \ γ	.01	.05	.1	.9	.95	.99
40	1.00	-2.58	-1.82	-1.41	1.39	1.80	2.62
	.75	-3.29	-2.25	-1.69	1.39	1.85	2.61
	.50	-6.21	-3.77	-2.91	1.63	2.16	2.96
60	1.00	-2.48	-1.78	-1.38	1.37	1.77	2.56
	.75	-3.22	-2.16	-1.68	1.42	1.84	2.66
	.50	-5.37	-3.56	-2.69	1.67	2.18	3.01
80	1.00	-2.51	-1.76	-1.37	1.37	1.76	2.49
	.75	-3.11	-2.10	-1.61	1.43	1.85	2.65
	.50	-5.14	-3.45	-2.62	1.71	2.16	3.08
100	1.00	-2.45	-1.74	-1.37	1.35	1.73	2.50
	.75	-3.12	-2.09	-1.60	1.44	1.85	2.61
	.50	-4.92	-3.34	-2.49	1.78	2.26	3.19
120	1.00	-2.44	-1.73	-1.35	1.35	1.74	2.48
	.75	-3.01	-2.01	-1.58	1.45	1.86	2.63
	.50	-4.50	-3.17	-2.44	1.75	2.27	3.13
∞	1.00	-2.45	-1.73	-1.35	1.35	1.73	2.45
	.75	-2.69	-1.90	-1.48	1.48	1.90	2.69
	.50	-3.69	-2.61	-2.03	2.03	2.61	3.69

TABLE 3 BIAS IN $\hat{R}(t)$

	25% Censoring					50% Censoring				
R(t)	40	60	80	100	120	40	60	80	100	120
.75	.005	.003	.003	.001	.002	.002	.000	.001	.000	.000
.80	.005	.003	.003	.002	.002	.004	.002	.002	.001	.001
.85	.004	.002	.003	.001	.001	.005	.002	.002	.001	.002
.90	.003	.001	.002	.001	.001	.004	.002	.002	.001	.001
.925	.002	.000	.001	.000	.001	.003	.001	.002	.001	.001
.95	.001	.000	.001	.000	.000	.001	.001	.001	.001	.001
.96	.000	-.000	.000	-.000	.000	.001	.000	.001	.000	.000
.97	-.000	-.000	.000	-.000	-.000	.000	.000	.000	.000	.000
.98	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000
.99	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000
.995	-.001	-.000	-.000	-.000	-.000	-.001	-.000	-.000	-.000	-.000
.996	-.001	-.000	-.000	-.000	-.000	-.001	-.000	-.000	-.000	-.000
.997	-.000	-.000	-.000	-.000	-.000	-.001	-.000	-.000	-.000	-.000
.998	-.000	-.000	-.000	-.000	-.000	-.001	-.000	-.000	-.000	-.000
.999	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000

TABLE 4 VARIANCE OF $\hat{R}(t)$

	25% Censoring					50% Censoring				
R(t)	n 40	60	80	100	120	40	60	80	100	120
.75	.0035	.0023	.0018	.0013	.0012	.0040	.0024	.0019	.0014	.0013
.80	.0030	.0019	.0015	.0011	.0010	.0032	.0020	.0016	.0012	.0011
.85	.0023	.0015	.0012	.0009	.0008	.0025	.0016	.0013	.0010	.0008
.90	.0015	.0010	.0008	.0006	.0006	.0016	.0011	.0009	.0007	.0006
.925	.0011	.0008	.0006	.0005	.0004	.0012	.0008	.0007	.0005	.0005
.95	.0007	.0005	.0004	.0003	.0003	.0008	.0006	.0004	.0004	.0003
.96	.0005	.0004	.0003	.0003	.0002	.0006	.0004	.0004	.0003	.0003
.97	.0004	.0003	.0003	.0002	.0002	.0005	.0003	.0003	.0003	.0002
.98	.0003	.0002	.0002	.0002	.0002	.0003	.0002	.0002	.0002	.0002
.99	.0002	.0002	.0001	.0001	.0001	.0002	.0002	.0002	.0001	.0001
.995	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
.996	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
.997	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
.998	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
.999	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

TABLE 5 COMPARISON OF $V(\hat{R}(t))$ WITH CRLB VARIANCE x 10^4 (n = 40)

R(t)	.75	.80	.85	.90	.925	.95	.96	.97	.98	.99	>.99
r = 30											
$V(\hat{R}(t))$	35	30	23	15	11	07	05	04	03	02	01
CRLB	33	28	22	14	10	06	04	03	02	01	01
r = 20											
$V(\hat{R}(t))$	40	32	25	16	12	08	06	05	03	02	01
CRLB	35	29	23	16	12	07	05	04	02	01	01

TABLE 6 A

90% LOWER CONFIDENCE LIMITS ON R(t)

	25% CENSORING					50% CENSORING				
$\hat{R}(t)$	n 40	60	80	100	120	40	60	80	100	120
.70	.623	.638	.641	.650	.654	.616	.639	.644	.652	.655
.72	.641	.657	.661	.669	.673	.635	.658	.663	.672	.674
.74	.659	.676	.681	.690	.693	.653	.677	.683	.691	.694
.76	.678	.696	.702	.710	.713	.674	.696	.703	.711	.714
.78	.698	.716	.723	.731	.734	.694	.716	.723	.732	.734
.80	.718	.737	.744	.752	.755	.715	.736	.744	.752	.755
.82	.739	.758	.766	.774	.776	.737	.757	.765	.773	.776
.84	.761	.780	.789	.796	.798	.759	.779	.787	.795	.797
.86	.783	.802	.810	.818	.821	.783	.801	.810	.817	.819
.88	.807	.826	.833	.841	.843	.807	.824	.832	.839	.842
.90	.832	.850	.857	.864	.866	.832	.847	.855	.862	.864
.92	.858	.875	.882	.888	.890	.858	.872	.879	.886	.888
.94	.886	.901	.907	.913	.914	.886	.898	.904	.910	.912
.95	.901	.915	.920	.925	.927	.901	.911	.917	.922	.924
.96	.917	.929	.934	.939	.940	.917	.925	.930	.935	.937
.97	.938	.943	.947	.952	.953	.933	.940	.944	.949	.951
.98	.951	.959	.963	.966	.967	.951	.956	.959	.964	.965
.99	.971	.976	.979	.981	.982	.971	.974	.977	.979	.980
.9925	.977	.981	.984	.986	.986	.977	.979	.982	.984	.985
.995	.983	.987	.989	.990	.990	.983	.985	.987	.988	.989
.996	.986	.989	.990	.992	.992	.986	.987	.989	.990	.991
.997	.989	.992	.993	.994	.994	.989	.989	.991	.992	.993
.998	.992	.994	.995	.995	.996	.992	.992	.994	.995	.995
.9985	.993	.995	.996	.996	.997	.993	.994	.995	.996	.996
.999	.994	.996	.997	.998	.998	.994	.995	.996	.997	.997

TABLE 6B

95% LOWER CONFIDENCE LIMITS ON R(t)

	25% CENSORING					50% CENSORING				
$\hat{R}(t)$	n 40	60	80	100	120	40	60	80	100	120
.70	.594	.626	.624	.625	.643	.600	.623	.628	.639	.646
.72	.613	.644	.643	.647	.662	.614	.641	.647	.659	.664
.74	.632	.662	.664	.669	.681	.632	.660	.667	.678	.683
.76	.651	.680	.684	.691	.701	.651	.679	.686	.698	.702
.78	.671	.699	.705	.713	.722	.671	.699	.707	.719	.722
.80	.692	.719	.726	.736	.743	.691	.719	.727	.741	.742
.82	.714	.740	.748	.759	.764	.712	.740	.749	.761	.762
.84	.737	.761	.771	.782	.786	.734	.761	.771	.782	.784
.86	.760	.784	.795	.806	.809	.757	.784	.793	.805	.806
.88	.785	.808	.819	.830	.832	.781	.807	.817	.827	.829
.90	.811	.833	.844	.854	.856	.807	.831	.841	.851	.852
.92	.839	.860	.870	.879	.881	.834	.857	.866	.876	.877
.94	.869	.888	.897	.904	.906	.863	.883	.892	.902	.903
.95	.885	.903	.911	.917	.920	.878	.879	.906	.915	.917
.96	.902	.919	.926	.932	.933	.894	.913	.920	.929	.931
.97	.920	.935	.941	.946	.948	.913	.929	.936	.946	.947
.98	.940	.953	.957	.962	.963	.933	.947	.952	.960	.961
.99	.964	.972	.976	.978	.979	.957	.968	.971	.975	.977
.9925	.970	.978	.981	.983	.984	.965	.974	.977	.980	.981
.995	.978	.984	.986	.988	.988	.973	.980	.983	.985	.986
.996	.981	.986	.988	.990	.990	.976	.983	.985	.988	.989
.997	.985	.989	.991	.992	.992	.980	.986	.988	.990	.991
.998	.988	.992	.993	.994	.995	.985	.990	.992	.993	.994
.9985	.991	.994	.995	.996	.996	.987	.992	.993	.994	.995
.999	.993	.995	.996	.997	.997	.990	.994	.995	.996	.996

TABLE 6C

98% LOWER CONFIDENCE LIMITS ON R(t)

	25% CENSORING					50% CENSORING				
$\hat{R}(t)$	n 40	60	80	100	120	40	60	80	100	120
.70	.571	[illegible]	.610	.622	.625	.590	.606	.613	.628	.631
.72	.589	.615	.629	.641	.645	.606	.623	.632	.647	.649
.74	.601	.635	.648	.661	.666	.622	.641	.650	.665	.669
.76	.627	.654	.668	.681	.687	.639	.660	.670	.685	.689
.78	.647	.675	.688	.702	.708	.657	.679	.689	.704	.708
.80	.669	.696	.709	.723	.730	.676	.698	.710	.725	.729
.82	.690	.718	.731	.745	.752	.695	.719	.731	.746	.754
.84	.713	.741	.753	.768	.775	.716	.740	.752	.768	.773
.86	.737	.764	.777	.791	.798	.737	.763	.775	.790	.792
.88	.763	.789	.801	.815	.822	.760	.786	.799	.813	.819
.90	.790	.816	.827	.840	.846	.785	.811	.824	.837	.848
.92	.819	.844	.854	.866	.871	.812	.838	.850	.863	.871
.94	.851	.873	.883	.893	.896	.842	.866	.877	.889	.893
.95	.868	.889	.898	.908	.911	.858	.881	.892	.903	.906
.96	.886	.906	.914	.923	.926	.875	.897	.907	.918	.922
.97	.906	.924	.931	.938	.941	.895	.915	.924	.934	.937
.98	.928	.943	.950	.955	.957	.917	.935	.943	.951	.953
.99	.954	.966	.970	.974	.976	.945	.959	.964	.971	.972
.9925	.962	.972	.976	.980	.981	.953	.966	.970	.976	.978
.995	.971	.979	.983	.985	.986	.963	.974	.977	.982	.984
.996	.975	.983	.985	.988	.989	.967	.978	.981	.985	.987
.997	.979	.986	.988	.990	.991	.972	.982	.984	.988	.989
.998	.984	.990	.991	.993	.994	.978	.986	.988	.991	.992
.9985	.987	.992	.993	.994	.995	.982	.989	.990	.993	.994
.999	.990	.994	.995	.996	.996	.986	.992	.993	.995	.995

TABLE 6D

99% LOWER CONFIDENCE LIMITS ON R(t)

	25% CENSORING					50% CENSORING				
$\hat{R}(t)$	n 40	60	80	100	120	40	60	80	100	120
.70	.555	.585	.601	.618	.623	.566	.590	.609	.613	.615
.72	.574	.603	.620	.636	.641	.582	.607	.626	.633	.636
.74	.592	.622	.638	.655	.661	.599	.624	.643	.652	.656
.76	.612	.642	.658	.674	.680	.617	.643	.661	.672	.677
.78	.632	.662	.678	.694	.701	.636	.662	.679	.693	.698
.80	.652	.684	.699	.715	.721	.655	.681	.698	.714	.720
.82	.674	.705	.720	.736	.743	.675	.702	.718	.735	.742
.84	.697	.728	.743	.759	.765	.696	.723	.739	.758	.765
.86	.722	.752	.766	.782	.788	.718	.746	.761	.780	.788
.88	.747	.777	.791	.806	.812	.742	.770	.784	.804	.812
.90	.775	.804	.816	.831	.837	.768	.796	.809	.829	.836
.92	.805	.832	.844	.858	.863	.795	.823	.836	.854	.861
.94	.838	.863	.873	.886	.890	.826	.853	.865	.881	.886
.95	.855	.879	.889	.901	.905	.843	.869	.881	.895	.899
.96	.874	.896	.906	.916	.920	.861	.886	.897	.910	.914
.97	.895	.915	.923	.933	.936	.881	.905	.915	.927	.931
.98	.918	.936	.943	.951	.954	.904	.926	.935	.945	.949
.99	.947	.960	.966	.971	.973	.934	.953	.959	.965	.967
.9925	.956	.968	.973	.977	.978	.943	.960	.966	.971	.974
.995	.966	.975	.980	.984	.984	.954	.969	.977	.978	.982
.996	.970	.979	.983	.986	.987	.959	.973	.978	.981	.985
.997	.975	.983	.986	.989	.989	.966	.977	.982	.985	.988
.998	.981	.987	.990	.992	.992	.974	.982	.986	.988	.991
.9985	.984	.990	.992	.993	.994	.978	.975	.988	.991	.993
.999	.987	.992	.994	.995	.996	.983	.988	.991	.993	.994

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